PRACTICE FINAL (VOJTA) - SOLUTIONS

PEYAM RYAN TABRIZIAN

(1) (a)
1) Let $y = (e^x + x)^{\frac{1}{x}}$
2) $\ln(y) = \frac{\ln(e^x+x)}{x}$
3) $\lim_{x\to 0} \ln(y) \stackrel{H}{=} \lim_{x\to 0} \frac{e^x+1}{e^x+x} = 2$
4) Hence $\lim_{x\to 0} y = \boxed{e^2}$

(b)

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan^2 x - 3}{7\tan^2 x + \sin x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1 - \frac{3}{\tan^2 x}}{7 + \frac{\sin x}{\tan^2 x}} = \frac{1 - \frac{3}{\infty}}{7 + \frac{0}{\infty}} = \frac{1}{7}$$

(c)

$$\lim_{x\to-\infty} \frac{3x+2}{\sqrt{x^2+7}} = \lim_{x\to-\infty} \frac{x(3+\frac{2}{x})}{\sqrt{x^2}\sqrt{1+\frac{7}{x^2}}} = \lim_{x\to-\infty} \frac{x(3+\frac{2}{x})}{|x|\sqrt{1+\frac{7}{x^2}}} = \lim_{x\to-\infty} \frac{x(3+\frac{2}{x})}{-x\sqrt{1+\frac{7}{x^2}}} = -3$$

(2) Notice that $0 \leq \frac{1}{x+\frac{1}{\ln(x)}} \leq \frac{1}{x}$ because $\frac{1}{\ln(x)} > 0$ when $x > 1$. Moreover, $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \frac{1}{x} = 0$, so by the **squeeze theorem**, $\lim_{x\to\infty} \frac{1}{x+\frac{1}{\ln(x)}} = 0$

(3) (a) Don't worry about this, no $\cosh$ on the exam!
(b) Again, don't worry, butI would still try to do it, because it's a good exercise in differentiating! The only thing you should know is that $\cosh' = \sinh$:

$$\frac{d}{dx}\frac{\ln(\cosh x^2)}{e^x+2} = \frac{\frac{(\sinh x^2)(2x)}{\cosh x^2}(e^x+2) - \ln(\cosh x^2)(e^x)}{(e^x+2)^2}$$

(4) (a) Using implicit differentiation: $3y^2y' = 4x^3 + 8y'$, now plugging in $x = 1$ and $y = 2$, you get: $12y' = 4 + 8y'$, so $4y' = 4$, so $\boxed{y' = 1}$
(b) Starting with $3y^2y' = 4x^3 + 8y'$, differentiate this again, and you get $6y(y')^2 + 3y^2y'' = 12x^2 + 8y''$, now plugging in $x = 1$, $y = 2$, and $y' = 1$ (which you got from (a)), you get: $12 + 12y'' = 12 + 8y''$, so $4y'' = 0$, so $\boxed{y'' = 0}$

Date: Tuesday, May 3rd, 2011.

(5) Here $f(x) = \tan(x)$, $a = \frac{\pi}{4}$, so:

$$L(x) = f(a) + f'(a)(x-a) = \tan(\frac{\pi}{4}) + \sec^2(\frac{\pi}{4})(x - \frac{\pi}{4}) = 1 + 2(x - \frac{\pi}{4})$$

Hence:

$$\tan(\frac{\pi}{4} + 0.01) \approx L(\frac{\pi}{4} + 0.01) = 1 + 2(\frac{\pi}{4} + 0.01 - \frac{\pi}{4}) = 1.02$$

(6) 1) Want to find: $\frac{dr}{dt}$
2) $V = \pi r^2 h$
3) $\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$
4) $-2\pi = 2\pi r \frac{dr}{dt}(6) + \pi r^2(-1)$, but from $V = \pi r^2 h$, and $V = 24\pi$ and $h = 6$, you get $r = 4$, and so: $-2\pi = 2\pi(2)\frac{dr}{dt}(6) + \pi(4)(-1)$, so $-2\pi = 24\pi\frac{dr}{dt} - 4\pi$, so $\boxed{\frac{dr}{dt} = \frac{1}{12}}$, so the radius is increasing by $\frac{1}{12}$ cm/sec

(7) 1) Endpoints: $f(-1) = -1 + 4 = 3$, $f(27) = 27 - 3(9) = 0$
2) Critical points: $f'(x) = 1 - 2x^{-\frac{1}{3}} = 0 \Leftrightarrow x = 8$, and $f(8) = 8 - 3(4) = -4$, also 0 is a critical point ($f'(0)$ is not defined), and $f(0) = 0$
3) Hence the absolute maximum is $\boxed{f(-1) = 3}$ and the absolute minimum is $\boxed{f(8) = -4}$

(8) Don't worry about slant asymptotes!

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to\infty} \frac{x^2(x - \frac{1}{x^2})}{x^2(1 - \frac{1}{x^2})} = \lim_{x\to\infty} \frac{x - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \infty$$

And similarly $\lim_{x\to-\infty} f(x) = -\infty$, so f has $\boxed{\text{no horizontal asymptotes}}$.

For the vertical asymptotes, the only candidates are $x = \pm 1$ (those are the values where f is undefined)

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1^+} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x\to 1^+} \frac{x^2 + x + 1}{x + 1} = \lim_{x\to 1^+} \frac{3}{2}$$

Similarly $\lim_{x\to 1^-} f(x) = \frac{3}{2}$, so $x = 1$ is **NOT** a vertical asymptote of f! (Note that you could also use l'Hopital's rule for this, but great care must be taken, it only works once! If you do it twice, you're in trouble!).

However,

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to(-1)^+} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x\to(-1)^+} \frac{x^2 + x + 1}{x + 1} = \lim_{x\to 1^+} \frac{3}{0^+} = \infty$$

Hence $\boxed{x = -1}$ is a vertical asymptote of f (no need to calculate $\lim_{x\to(-1)^-} f(x)$, since you already found ∞ for one side of the limit! Also, again, L'Hopital's rule works perfectly well, but you can only use it once!)

(9) We have $a(t) = -3$, so $v(t) = -3t + C$. But $v(0) = C = 30$, so $v(t) = -3t + 30$, $v(t) = 0 \Leftrightarrow -3t + 30 = 0 \Leftrightarrow \boxed{t = 10 \text{ sec}}$

(10) Ignore this, Newton's method is not on the exam!

(11) (a) $\boxed{0}$ (odd function!)

(b) Notice that $9x^2 = (3x)^2$, now let $u = 3x$, then $du = 3dx$, so $dx = \frac{1}{3}du$, and:

$$\int \frac{dx}{9x^2+1} = \int \frac{\frac{1}{3}du}{u^2+1} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\tan^{-1}(3x) + C$$

(c)

$$\int \frac{x^4+1}{x^3}dx = \int \frac{x^4}{x^3} + \frac{1}{x^3}dx = \int x + x^{-3}dx = \frac{x^2}{2} + \frac{x^{-2}}{-2} + C = \frac{x^2}{2} - \frac{1}{2x^2} + C$$

(d) Let $u = \cos(x^2)$, then $du = -2x\sin(x^2)$, so:

$$\int (x\sin(x^2))e^{\cos(x^2)}dx = \int -\frac{1}{2}e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos(x^2)} + C$$

(e) Let $u = x - 1$, then $du = dx$, $x = u + 1$, and $u(1) = 0$, $u(2) = 1$, so:

$$\int_1^2 x\sqrt[4]{x-1}dx = \int_0^1 (u+1)\sqrt[4]{u}du = \int_0^1 u^{\frac{5}{4}} + u^{\frac{1}{4}}du = \left[\frac{4}{9}u^{\frac{9}{4}} + \frac{4}{5}u^{\frac{5}{4}}\right]_0^1 = \frac{4}{9} + \frac{4}{5} = \frac{56}{45}$$

(12) $f(x) = x^2$, $a = 1$, $b = 2$, $\Delta x = \frac{2-1}{n} = \frac{1}{n}$, $x_i = a + (\Delta x)i = 1 + \frac{i}{n}$, so:

$$
\begin{aligned}
\int_1^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(1+\frac{i}{n}\right)^2 \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n (\frac{1}{n} + \frac{2i}{n^2} + \frac{i^2}{n^3}) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n 1 + \frac{2}{n^2}\sum_{i=1}^n i + \frac{1}{n^3}\sum_{i=1}^n i^2 \\
&= \lim_{n\to\infty} \frac{1}{n}n + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} 1 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} \\
&=1+1+\frac{2}{6} \\
&=\frac{7}{3}
\end{aligned}
$$

(13) (a) If you draw a picture, you should notice that the disk method doesn't work (region is not glued to the $y-$axis), and the washer method would be a pain, because you'd have to calculate x in terms of y! So the only hope that's left is to use the **shell method**!
$k = 0$, Radius = $|x-0| = x$ (since $x \geq 0$), Height = Bigger - Smaller $= 3x - x^2 - 2x^2 = 3x - 3x^2$
Points of intersection: $2x^2 = 3x - x^2 \Leftrightarrow 3x^2 = 3x \Leftrightarrow x = 0 \text{or} x = 1$

$$
V = \int_0^1 2\pi(x)(3x-x^2)dx = \int_0^1 2\pi(3x^2-3x^3)dx = 2\pi\left[x^3 - \frac{3}{4}x^4\right]_0^1 = \frac{2\pi}{4} = \frac{\pi}{2}
$$

(b) Washer method (vertical washers):
$k = 1$, Outer = Bigger $-k = 3x - x^2 + 1$, Inner = Smaller $-k = 2x^2 + 1$,
Points of intersection $= 0, 1$ (as in (a)), so:

$$
V = \int_0^1 \pi((3x - x^2 + 1)^2 - (2x^2+1)^2)dx
$$